

## Complete Solutions to Miscellaneous Exercise 1

1. (i) How do we find  $(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B})$  given  $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$  and  $\mathbf{B} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$ ?

Evaluate  $\mathbf{A} - \mathbf{B}$ ,  $\mathbf{A} + \mathbf{B}$  and then multiply them together.

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} -4 & -4 \\ -4 & -4 \end{pmatrix} = -4 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix} = 2 \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}$$

Multiplying these together gives

$$\begin{aligned} (\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B}) &= -4 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} 2 \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} \\ &= -8 \begin{pmatrix} 8 & 10 \\ 8 & 10 \end{pmatrix} = -16 \begin{pmatrix} 4 & 5 \\ 4 & 5 \end{pmatrix} \end{aligned}$$

(ii) How do we determine  $\mathbf{A}^2 - \mathbf{B}^2$ ?

$$\begin{aligned} \mathbf{A}^2 - \mathbf{B}^2 &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^2 - \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}^2 \\ &= \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} - \begin{pmatrix} 67 & 78 \\ 91 & 106 \end{pmatrix} \\ &= \begin{pmatrix} -60 & -68 \\ -76 & -84 \end{pmatrix} = -4 \begin{pmatrix} 15 & 17 \\ 19 & 21 \end{pmatrix} \end{aligned}$$

$\mathbf{A}^2 - \mathbf{B}^2 \neq (\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B})$  because

$$(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B}) = \mathbf{A}^2 + \underbrace{\mathbf{AB} - \mathbf{BA}}_{\neq \mathbf{0} \text{ because } \mathbf{AB} \neq \mathbf{BA}} - \mathbf{B}^2$$

Remember matrix multiplication is **not** commutative, that is

$$\mathbf{AB} \neq \mathbf{BA}$$

so we **cannot** have  $\mathbf{A}^2 - \mathbf{B}^2 = (\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B})$ .

2. We have an error in the first line:

$$\mathbf{AB}(\mathbf{AB})^{-1} = \mathbf{ABA}^{-1}\mathbf{B}^{-1}$$

because  $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \neq \mathbf{A}^{-1}\mathbf{B}^{-1}$  so the above are **not** equal. Again there is an error in the second line of the derivation:

$$\mathbf{ABA}^{-1}\mathbf{B}^{-1} = \mathbf{AA}^{-1}\mathbf{BB}^{-1}$$

because  $\mathbf{BA}^{-1} \neq \mathbf{A}^{-1}\mathbf{B}$  [Not Equal]. Matrix multiplication is **not** commutative.

3. We are given the matrix  $\mathbf{A} = \frac{1}{7} \begin{pmatrix} 3 & -2 & -6 \\ -2 & 6 & -3 \\ -6 & -3 & -2 \end{pmatrix}$ .

(a) The matrix  $\mathbf{A}^2$  is evaluated by

$$\begin{aligned}
\mathbf{A}^2 &= \frac{1}{7} \begin{pmatrix} 3 & -2 & -6 \\ -2 & 6 & -3 \\ -6 & -3 & -2 \end{pmatrix} \frac{1}{7} \begin{pmatrix} 3 & -2 & -6 \\ -2 & 6 & -3 \\ -6 & -3 & -2 \end{pmatrix} \\
&= \frac{1}{7} \times \frac{1}{7} \begin{pmatrix} 3 & -2 & -6 \\ -2 & 6 & -3 \\ -6 & -3 & -2 \end{pmatrix} \begin{pmatrix} 3 & -2 & -6 \\ -2 & 6 & -3 \\ -6 & -3 & -2 \end{pmatrix} \\
&= \frac{1}{49} \begin{pmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{I}
\end{aligned}$$

Hence  $\mathbf{A}^2 = \mathbf{I}$ . What is  $\mathbf{A}^3$  equal to?

$$\mathbf{A}^3 = \mathbf{A}^2 \mathbf{A} = \mathbf{I} \mathbf{A} = \mathbf{A} = \frac{1}{7} \begin{pmatrix} 3 & -2 & -6 \\ -2 & 6 & -3 \\ -6 & -3 & -2 \end{pmatrix}$$

(b) Since we have  $\mathbf{A}^2 = \mathbf{I}$  therefore  $\mathbf{A}^{-1} = \mathbf{A}$  and

$$\mathbf{A}^{2004} = (\mathbf{A}^2)^{1002} = \mathbf{I}^{1002} = \mathbf{I}$$

where  $\mathbf{I}$  is the identity 3 by 3 matrix.

4. We have the following:

$$\mathbf{A}^t = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}^t = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

The matrix multiplication  $\mathbf{AB}$  is **not** valid because the number of columns (3) of matrix  $\mathbf{A}$  does **not** equal the number of rows (4) of matrix  $\mathbf{B}$ .

The matrix addition  $\mathbf{B} + \mathbf{C}$  is **not** valid because matrices  $\mathbf{B}$  and  $\mathbf{C}$  are different sizes.

The matrix subtraction  $\mathbf{A} - \mathbf{B}$  is **not** valid because matrices  $\mathbf{A}$  and  $\mathbf{B}$  are different sizes.

The matrix multiplication  $\mathbf{CB}$  is given by

$$\mathbf{CB} = \begin{pmatrix} 1 & -1 & 1 & 0 \\ -2 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -1 & 1 \end{pmatrix}$$

The matrix multiplication  $\mathbf{BC}^t$  is **not** valid because

$$\mathbf{BC}^t = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & 0 \\ -2 & 1 & 2 & 3 \end{pmatrix}^t = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 1 \\ 1 & 2 \\ 0 & 3 \end{pmatrix}$$

This matrix multiplication is impossible because the number of columns (2) of the matrix  $\mathbf{B}$  does **not** equal the number of rows (4) of the matrix  $\mathbf{C}^t$ .

The matrix multiplication  $\mathbf{A}^2$  is given by

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 0 & -2 & -4 \\ 3 & 3 & 3 \\ 6 & 8 & 10 \end{pmatrix}$$

5. We need to show that  $\mathbf{A}^T \mathbf{A}$  is symmetric for all matrices  $\mathbf{A}$ .

*Proof.*

First we need to establish that  $\mathbf{A}^T \mathbf{A}$  is a valid multiplication operation. Let matrix  $\mathbf{A}$  be of size  $m \times n$ . Then  $\mathbf{A}^T$  is of size  $n \times m$ . The multiplication operation  $\mathbf{A}^T \mathbf{A}$  is valid provided the number of columns of the left hand matrix is equal to the number of rows of the right hand matrix. *How many columns does  $\mathbf{A}^T$  have?*

*m. How many rows does  $\mathbf{A}$  have?*

*m. Hence the matrix multiplication  $\mathbf{A}^T \mathbf{A}$  is valid.*

*How do we show that  $\mathbf{A}^T \mathbf{A}$  is symmetric?*

We need to prove that  $(\mathbf{A}^T \mathbf{A})^T = \mathbf{A}^T \mathbf{A}$ . We have

$$\begin{aligned} (\mathbf{A}^T \mathbf{A})^T &= \mathbf{A}^T (\mathbf{A}^T)^T && \left[ \text{Because } (\mathbf{XY})^T = \mathbf{Y}^T \mathbf{X}^T \right] \\ &= \mathbf{A}^T \mathbf{A} && \left[ \text{Because } (\mathbf{X}^T)^T = \mathbf{X} \right] \end{aligned}$$

Hence  $\mathbf{A}^T \mathbf{A}$  is symmetric.

6. We need to find conditions on  $a, b, c$  and  $d$  so that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Expanding both of these separately we have

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} &= \begin{bmatrix} a+3b & 2a \\ c+3d & 2c \end{bmatrix} \\ \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} a+2c & b+2d \\ 3a & 3b \end{bmatrix} \end{aligned}$$

Equating each corresponding entry of the matrices on the right hand side gives:

$$\left. \begin{aligned} a+3b &= a+2c \\ 2a &= b+2d \\ c+3d &= 3a \\ 2c &= 3b \end{aligned} \right\} \text{ gives } a=c \text{ and } b=d=\frac{2}{3}c$$

7. We can find (many) example(s) such that  $2 \times 2$  matrix  $\mathbf{A}$  and  $2 \times 1$  nonzero vectors  $\mathbf{u}$  and  $\mathbf{v}$  such that  $\mathbf{A}\mathbf{u} = \mathbf{A}\mathbf{v}$  yet  $\mathbf{u} \neq \mathbf{v}$ .

Let  $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ ,  $\mathbf{u} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$  and  $\mathbf{v} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$  then

$$\mathbf{A}\mathbf{u} = \mathbf{A}\mathbf{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{O} \text{ but } \mathbf{u} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \mathbf{v}$$

8. (a) We are given the matrices  $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$  and  $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ :

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix}$$

$$\mathbf{B}^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -4 & 3 \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}$$

(b) We are given  $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and  $\mathbf{B} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$ :

$$\mathbf{AB} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix}$$

Subtracting each corresponding entry gives

$$\begin{aligned} \mathbf{AB} - \mathbf{BA} &= \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} - \begin{pmatrix} ae+cf & be+df \\ ag+ch & bg+dh \end{pmatrix} \\ &= \begin{pmatrix} bg-cf & af+bh-be-df \\ ce+dg-ag-ch & cf-bg \end{pmatrix} \end{aligned}$$

9. We need to evaluate the first four indices of the given matrix  $\mathbf{M} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ :

$$\mathbf{M}^2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 2\mathbf{M}$$

$$\begin{aligned} \mathbf{M}^3 &= \mathbf{M}^2 \mathbf{M} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} = 4 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 4\mathbf{M} \end{aligned}$$

$$\begin{aligned} \mathbf{M}^4 &= \mathbf{M}^3 \mathbf{M} = \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix} = 8 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 8\mathbf{M} \end{aligned}$$

We need to predict a formula for  $\mathbf{M}^n = c(n)\mathbf{M}$ . What do you notice about the above results?

$$\mathbf{M}^2 = 2\mathbf{M}, \mathbf{M}^3 = 4\mathbf{M} \text{ and } \mathbf{M}^4 = 8\mathbf{M}$$

The predicted formula is  $\mathbf{M}^n = 2^{n-1}\mathbf{M}$ . Thus  $c(n) = 2^{n-1}$ .

10. We are given  $\mathbf{A} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$  which we can rewrite as  $\mathbf{A} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ .

(i) We have

$$\begin{aligned} \mathbf{A}^2 &= \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{3^2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{3^2} \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \\ &= \frac{2}{3^2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

(ii) Similarly, we have

$$\begin{aligned} \mathbf{A}^3 &= \mathbf{A}^2 \mathbf{A} = \frac{1}{3^2} \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{3^3} \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} = \frac{2^2}{3^3} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \frac{2^3}{3^3} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \left( \frac{2}{3} \right)^3 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

To prove  $\mathbf{A}^n = \frac{1}{2} \left( \frac{2}{3} \right)^n \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$  we use mathematical induction.

*Proof.*

Clearly the result is true for  $n=1$  because we have

$$\mathbf{A} = \frac{1}{2} \left( \frac{2}{3} \right) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

We assume the given result is true for  $n=k$ , that is

$$\mathbf{A}^k = \frac{1}{2} \left( \frac{2}{3} \right)^k \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad (*)$$

Required to prove this for  $n=k+1$ :

$$\begin{aligned} \mathbf{A}^{k+1} &= \mathbf{A}^k \mathbf{A} = \underbrace{\frac{1}{2} \left( \frac{2}{3} \right)^k \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}_{\text{By } (*)} \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \frac{1}{3} \left( \frac{2}{3} \right)^k \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \frac{1}{3} \left( \frac{2}{3} \right)^k \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \frac{1}{2} \frac{2}{3} \left( \frac{2}{3} \right)^k \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \left( \frac{2}{3} \right)^{k+1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2} \left( \frac{2}{3} \right)^{k+1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

Hence by mathematical induction we have our result  $\mathbf{A}^n = \frac{1}{2} \left( \frac{2}{3} \right)^n \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ .

11. In general if  $\mathbf{B}$  is a  $m \times p$  matrix and  $\mathbf{C}$  is a  $p \times n$  matrix then  $\mathbf{BC}$  is a  $m \times n$  matrix. Since we are told that  $\mathbf{BC}$  is a  $4 \times 6$  matrix therefore the matrix  $\mathbf{B}$  has 4 rows.

12. (a) We are given the matrix  $\mathbf{A} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ . How can we check that this is invertible or not?

or not?

By carrying out row operations:

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Execute the following row operations  $2R_2 + R_1$  and  $2R_3 + R_1$ :

$$\begin{array}{l} R_1 \\ R'_2 = 2R_2 + R_1 \\ R'_3 = 2R_3 + R_1 \end{array} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

Adding the bottom two rows  $R'_3 + R'_2$  gives

$$\begin{array}{l} R_1 \\ R'_2 \\ R''_3 = R'_3 + R'_2 \end{array} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Since we have a row of zeros therefore we conclude that the matrix  $\mathbf{A}$  is **non-invertible (singular)**.

(b) We are given that  $\mathbf{ABC} = \mathbf{I}_n$ . Pre-multiplying this by  $\mathbf{A}^{-1}$  and post multiplying by  $\mathbf{C}^{-1}$  gives

$$\begin{aligned} \mathbf{A}^{-1}(\mathbf{ABC})\mathbf{C}^{-1} &= \mathbf{A}^{-1}\mathbf{I}_n\mathbf{C}^{-1} \\ \underbrace{(\mathbf{A}^{-1}\mathbf{A})}_{=\mathbf{I}} \underbrace{\mathbf{B}(\mathbf{CC}^{-1})}_{=\mathbf{I}} &= \mathbf{A}^{-1}\mathbf{C}^{-1} = (\mathbf{CA})^{-1} \\ \mathbf{B} &= (\mathbf{CA})^{-1} \end{aligned}$$

Since  $\mathbf{B} = (\mathbf{CA})^{-1}$  and we are given that  $\mathbf{C}$  and  $\mathbf{A}$  are invertible, so  $\mathbf{CA}$  is invertible and

$$\mathbf{B}^{-1} = ((\mathbf{CA})^{-1})^{-1} = \mathbf{CA} \quad \left[ \text{Because } [\mathbf{X}^{-1}]^{-1} = \mathbf{X} \right]$$

Hence matrix  $\mathbf{B}$  is invertible.

13. (a) An  $n \times n$  matrix  $\mathbf{A}$  is invertible if and only if there exists a  $n \times n$  matrix  $\mathbf{B}$  such that  $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$

The matrix  $\mathbf{B}$  is normally denoted by  $\mathbf{A}^{-1}$ .

(b) To find the inverse of the given matrix  $\mathbf{A} = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix}$  we use the Gaussian process

given by:

Step 1: Formulate the augmented matrix  $[\mathbf{A} \mid \mathbf{I}]$ .

Step 2: Use row operations to convert this  $[\mathbf{A} \mid \mathbf{I}]$  to  $[\mathbf{I} \mid \mathbf{B}]$ . If this is possible then  $\mathbf{B}$  is the inverse of the matrix  $\mathbf{A}$ .

We have  $[\mathbf{A} \mid \mathbf{I}]$  is given by

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \left[ \begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 7 & 6 & 0 & 1 & 0 \\ 2 & 7 & 7 & 0 & 0 & 1 \end{array} \right]$$

Carrying out the row operations  $R_2 - R_1$  and  $R_3 - R_1$ :

$$\begin{array}{l} R_1 \\ R'_2 = R_2 - R_1 \\ R'_3 = R_3 - R_1 \end{array} \left[ \begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array} \right]$$

Executing the row operation  $R'_3 - R'_2$ :

$$\begin{array}{l} R_1 \\ R'_2 \\ R''_3 = R'_3 - R'_2 \end{array} \left[ \begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right]$$

Executing the row operation  $R_1 - 6R'_2$ :

$$\begin{array}{l} R_1^* = R_1 - 6R'_2 \\ R'_2 \\ R''_3 \end{array} \left[ \begin{array}{ccc|ccc} 2 & 0 & 6 & 7 & -6 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right]$$

Executing the row operation  $R_1^* - 6R''_3$ :

$$\begin{array}{l} R_1^{**} = R_1^* - 6R''_3 \\ R'_2 \\ R''_3 \end{array} \left[ \begin{array}{ccc|ccc} 2 & 0 & 0 & 7 & 0 & -6 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right]$$

Dividing the top row by 2 gives us our result of  $[\mathbf{I} \mid \mathbf{B}]$ :

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 7/2 & 0 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array} \right]$$

Hence  $\mathbf{A}^{-1} = \begin{bmatrix} 7/2 & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ . We can check this is indeed the inverse of  $\mathbf{A}$  by carrying out

the matrix multiplication:

$$\mathbf{A}\mathbf{A}^{-1} = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 7 & 6 \\ 2 & 7 & 7 \end{bmatrix} \begin{bmatrix} 7/2 & 0 & -3 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}$$

14. We are given the linear equations

$$x + y + 2z = 8$$

$$-x - 2y + 3z = 1$$

$$3x - 7y + 4z = 10$$

We can write these in matrix form as

$$\begin{pmatrix} 1 & 1 & 2 \\ -1 & -2 & 3 \\ 3 & -7 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \\ 10 \end{pmatrix}$$

We write this as an augmented matrix and then carry out row operations.

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \left( \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{array} \right)$$

Executing  $R_2 + R_1$  and  $R_3 - 3R_1$ :

$$\begin{array}{l} R_1 \\ R_2^* = R_2 + R_1 \\ R_3^* = R_3 - 3R_1 \end{array} \left( \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & -10 & -2 & -14 \end{array} \right)$$

Carrying out the row operation  $R_3^* - 10R_2^*$ :

$$\begin{array}{l} R_1 \\ R_2^* \\ R_3^{**} = R_3^* - 10R_2^* \end{array} \left( \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & -1 & 5 & 9 \\ 0 & 0 & -52 & -104 \end{array} \right)$$

Dividing the bottom row by  $-52$  and the middle row by  $-1$  gives

$$\begin{array}{l} R_1 \\ R_2^\dagger \\ R_3^\dagger \end{array} \left( \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & -5 & -9 \\ 0 & 0 & 1 & 2 \end{array} \right)$$

Executing  $R_2^\dagger + 5R_3^\dagger$  gives

$$\begin{array}{l} R_1 \\ R_2^{\dagger\dagger} = R_2^\dagger + 5R_3^\dagger \\ R_3^\dagger \end{array} \left( \begin{array}{ccc|c} 1 & 1 & 2 & 8 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right)$$

Carrying out the row operation  $R_1 - R_2^{\dagger\dagger}$ :

$$\begin{array}{l} R_1^* = R_1 - R_2^{\dagger\dagger} \\ R_2^{\dagger\dagger} \\ R_3^\dagger \end{array} \left( \begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right)$$

Carrying out the row operation  $R_1^* - 2R_3^\dagger$

$$\begin{array}{l} R_1^{**} = R_1^* - 2R_3^\dagger \\ R_2^{\dagger\dagger} \\ R_3^\dagger \end{array} \begin{array}{c} x \quad y \quad z \\ \left( \begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array} \right) \end{array}$$

Hence we have the solution  $x = 3$ ,  $y = 1$  and  $z = 2$ .

Check by substituting these into the given equations:

$$3 + 1 + 2(2) = 8$$

$$-3 - 2(1) + 3(2) = 1$$

$$3(3) - 7(1) + 4(2) = 10$$



15. (a) The linear system  $\mathbf{Ax} = \mathbf{b}$  where  $\mathbf{A} = \begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix}$ ,  $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$  and  $\mathbf{b} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$  has it's

augmented matrix given by:

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \left[ \begin{array}{ccc|c} 1 & 2 & 7 & 3 \\ -2 & 5 & 4 & 3 \\ -5 & 6 & -3 & 1 \end{array} \right]$$

Carrying out the row operations  $R_2 + 2R_1$  and  $R_3 + 5R_1$ :

$$\begin{array}{l} R_1 \\ R_2^* = R_2 + 2R_1 \\ R_3^* = R_3 + 5R_1 \end{array} \left[ \begin{array}{ccc|c} 1 & 2 & 7 & 3 \\ 0 & 9 & 18 & 9 \\ 0 & 16 & 32 & 16 \end{array} \right]$$

Dividing the middle row by 9 and the bottom row by 16 gives

$$\begin{array}{l} R_1 \\ R_2^{**} \\ R_3^{**} \end{array} \left[ \begin{array}{ccc|c} 1 & 2 & 7 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{array} \right]$$

Subtracting the bottom two rows  $R_3^{**} - R_2^{**}$  gives

$$\begin{array}{l} R_1 \\ R_2^{**} \\ R_3^+ \end{array} \left[ \begin{array}{ccc|c} 1 & 2 & 7 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

This is **not** in reduced row echelon form. *Why not?*

Because of the 2 in the top row. Remember any non-zero entry in a column containing a leading 1 is the leading 1. *How can we remove the 2?*

By executing the row operation  $R_1 - 2R_2^{**}$ :

$$\begin{array}{l} R_1 - 2R_2^{**} \\ R_2^{**} \\ R_3^+ \end{array} \begin{array}{c} x \quad y \quad z \\ \left[ \begin{array}{ccc|c} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$

This is now in reduced row echelon form.

(b) We have 2 non-zero rows and 3 unknowns so there is  $3 - 2 = 1$  free variable. *Which one is the free variable?*

z. Let  $z = t$  then from the middle row  $R_2^{**}$  we have

$$y + 2z = 1 \Rightarrow y = 1 - 2z = 1 - 2t$$

From the top row we have

$$x + 3z = 1 \Rightarrow x = 1 - 3z = 1 - 3t$$

Hence our solution set is  $x = 1 - 3t$ ,  $y = 1 - 2t$  and  $z = t$ . In parametric vector form we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 - 3t \\ 1 - 2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$

16. We are given the linear system:

$$\begin{aligned}x_1 - x_2 - 2x_3 - 8x_4 &= -3 \\ -2x_1 + x_2 + 2x_3 + 9x_4 &= 5 \\ 3x_1 - 2x_2 - 3x_3 - 15x_4 &= -9\end{aligned}$$

The augmented matrix is given by

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \left( \begin{array}{cccc|c} 1 & -1 & -2 & -8 & -3 \\ -2 & 1 & 2 & 9 & 5 \\ 3 & -2 & -3 & -15 & -9 \end{array} \right)$$

Executing the row operations  $R_2 + 2R_1$  and  $R_3 - 3R_1$ :

$$\begin{array}{l} R_1 \\ R_2' = R_2 + 2R_1 \\ R_3' = R_3 - 3R_1 \end{array} \left( \begin{array}{cccc|c} 1 & -1 & -2 & -8 & -3 \\ 0 & -1 & -2 & -7 & -1 \\ 0 & 1 & 3 & 9 & 0 \end{array} \right)$$

Adding the bottom 2 rows  $R_3' + R_2'$ :

$$\begin{array}{l} R_1 \\ R_2' \\ R_3'' = R_3' + R_2' \end{array} \left( \begin{array}{cccc|c} 1 & -1 & -2 & -8 & -3 \\ 0 & -1 & -2 & -7 & -1 \\ 0 & 0 & 1 & 2 & -1 \end{array} \right)$$

Carrying out the row operation  $R_1 - R_2'$ :

$$\begin{array}{l} R_1' = R_1 - R_2' \\ R_2' \\ R_3'' \end{array} \left( \begin{array}{cccc|c} 1 & 0 & 0 & -1 & -2 \\ 0 & -1 & -2 & -7 & -1 \\ 0 & 0 & 1 & 2 & -1 \end{array} \right)$$

Multiplying the middle row by  $-1$  gives

$$\begin{array}{l} R_1' \\ -R_2' \\ R_3'' \end{array} \left( \begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 2 & 7 & 1 \\ 0 & 0 & 1 & 2 & -1 \end{array} \right)$$

It is **not** in reduced row echelon form but we can find a solution set from this.

Since we have 4 unknowns and 3 non-zero rows therefore there is  $4 - 3 = 1$  free variable.

*Which variable is free?*

$x_4$  because it **does not** appear at the beginning of any row. Let  $x_4 = t$ . From the bottom row  $R_3''$  we have

$$x_3 + 2x_4 = -1 \text{ implies } x_3 = -1 - 2x_4 = -1 - 2t$$

From the middle row we have

$$\begin{aligned}x_2 + 2x_3 + 7x_4 &= 1 \text{ gives } x_2 = 1 - 2x_3 - 7x_4 \\ &= 1 - 2(-1 - 2t) - 7t = 3 - 3t\end{aligned}$$

From the top row we have

$$x_1 - x_4 = -2 \Rightarrow x_1 = -2 + x_4 = -2 + t$$

Our solution set is  $x_1 = -2 + t$ ,  $x_2 = 3 - 3t$ ,  $x_3 = -1 - 2t$  and  $x_4 = t$ . We can write this in vector form as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2+t \\ 3-3t \\ -1-2t \\ t \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ -2 \\ 1 \end{bmatrix}$$

17. The question says solve the given linear system by using Gauss-Jordan Elimination.  
*What does this mean?*

Means you have to place the augmented matrix in **reduced** row echelon form. We are given the linear system:

$$\begin{aligned} x_1 + 2x_2 + x_3 - 4x_4 &= 1 \\ x_1 + 3x_2 + 7x_3 + 2x_4 &= 2 \\ x_1 - 11x_3 - 16x_4 &= -1 \end{aligned}$$

The augmented matrix is

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \left( \begin{array}{cccc|c} 1 & 2 & 1 & -4 & 1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & 0 & -11 & -16 & -1 \end{array} \right)$$

Exchange rows  $R_1$  and  $R_3$ :

$$\begin{array}{l} R^*_1 = R_3 \\ R_2 \\ R^*_3 = R_1 \end{array} \left( \begin{array}{cccc|c} 1 & 0 & -11 & -16 & -1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & 2 & 1 & -4 & 1 \end{array} \right)$$

Executing the row operations  $R_2 - R^*_1$  and  $R^*_3 - R^*_1$ :

$$\begin{array}{l} R^*_1 \\ R^*_2 = R_2 - R^*_1 \\ R^{**}_3 = R^*_3 - R^*_1 \end{array} \left( \begin{array}{cccc|c} 1 & 0 & -11 & -16 & -1 \\ 0 & 3 & 18 & 18 & 3 \\ 0 & 2 & 12 & 12 & 2 \end{array} \right)$$

Dividing the middle row by 3 and the bottom row by 2 gives

$$\begin{array}{l} R^*_1 \\ R^\dagger_2 \\ R^\dagger_3 \end{array} \left( \begin{array}{cccc|c} 1 & 0 & -11 & -16 & -1 \\ 0 & 1 & 6 & 6 & 1 \\ 0 & 1 & 6 & 6 & 1 \end{array} \right)$$

Carrying out the row operation  $R^\dagger_3 - R^\dagger_2$ :

$$\begin{array}{l} R^*_1 \\ R^\dagger_2 \\ R^{\dagger\dagger}_3 = R^\dagger_3 - R^\dagger_2 \end{array} \begin{array}{c} x_1 \quad x_2 \quad x_3 \quad x_4 \\ \left( \begin{array}{cccc|c} 1 & 0 & -11 & -16 & -1 \\ 0 & 1 & 6 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right) \end{array}$$

We have 2 free variables. *Which of these variables are free?*

$x_3$  and  $x_4$ . Let  $x_3 = s$  and  $x_4 = t$ . From the middle row  $R^\dagger_2$  we have

$$x_2 + 6x_3 + 6x_4 = 1 \Rightarrow x_2 = 1 - 6x_3 - 6x_4 = 1 - 6s - 6t$$

From the top row we have

$$x_1 - 11x_3 - 16x_4 = -1 \Rightarrow x_1 = -1 + 11x_3 + 16x_4 = -1 + 11s + 16t$$

Our solution set is  $x_1 = -1 + 11s + 16t$ ,  $x_2 = 1 - 6s - 6t$ ,  $x_3 = s$  and  $x_4 = t$ . In vector

form we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1+11s+16t \\ 1-6s-6t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 11 \\ -6 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 16 \\ -6 \\ 0 \\ 1 \end{bmatrix}$$

18. We are given

$$\begin{array}{l} R_1 \begin{bmatrix} 1 & 2 & 5 & 6 \end{bmatrix} \\ R_2 \begin{bmatrix} 3 & 7 & 3 & 2 \end{bmatrix} \\ R_3 \begin{bmatrix} 2 & 5 & 11 & 22 \end{bmatrix} \end{array}$$

(i) Carrying out the row operations  $R_2 - 3R_1$  and  $R_3 - 2R_1$ :

$$\begin{array}{l} R_1 \begin{bmatrix} 1 & 2 & 5 & 6 \end{bmatrix} \\ R_2^* = R_2 - 3R_1 \begin{bmatrix} 0 & 1 & -12 & -16 \end{bmatrix} \\ R_3^* = R_3 - 2R_1 \begin{bmatrix} 0 & 1 & 1 & 10 \end{bmatrix} \end{array}$$

Executing the row operation  $R_3^* - R_2^*$  gives

$$\begin{array}{l} R_1 \begin{bmatrix} 1 & 2 & 5 & 6 \end{bmatrix} \\ R_2^* \begin{bmatrix} 0 & 1 & -12 & -16 \end{bmatrix} \\ R_3^{**} = R_3^* - R_2^* \begin{bmatrix} 0 & 0 & 13 & 26 \end{bmatrix} \end{array}$$

This is now in echelon form.

(ii) What do we need to do in order to place the above matrix into row echelon form?

Divide the bottom row  $R_3^{**}$  by 13:

$$\begin{array}{l} R_1 \begin{bmatrix} 1 & 2 & 5 & 6 \end{bmatrix} \\ R_2^* \begin{bmatrix} 0 & 1 & -12 & -16 \end{bmatrix} \\ R_3^{***} = R_3^{**}/13 \begin{bmatrix} 0 & 0 & 1 & 2 \end{bmatrix} \end{array}$$

(iii) For reduced row echelon form we need to ensure that the only non-zero entry containing a leading zero is that leading zero. What does this mean in this case?

Need to convert the 2, 5 and -12 to zero.

Carrying out the row operation  $R_2^* + 12R_3^{***}$ :

$$\begin{array}{l} R_1 \begin{bmatrix} 1 & 2 & 5 & 6 \end{bmatrix} \\ R_2^{**} = R_2^* + 12R_3^{***} \begin{bmatrix} 0 & 1 & 0 & 8 \end{bmatrix} \\ R_3^{***} \begin{bmatrix} 0 & 0 & 1 & 2 \end{bmatrix} \end{array}$$

Executing the row operations  $R_1 - 2R_2^{**}$ :

$$\begin{array}{l} R_1^* = R_1 - 2R_2^{**} \begin{bmatrix} 1 & 0 & 5 & -10 \end{bmatrix} \\ R_2^{**} \begin{bmatrix} 0 & 1 & 0 & 8 \end{bmatrix} \\ R_3^{***} \begin{bmatrix} 0 & 0 & 1 & 2 \end{bmatrix} \end{array}$$

The last row operation is  $R_1^* - 5R_3^{***}$ :

$$\begin{array}{l} R_1^{**} = R_1^* - 5R_3^{***} \begin{bmatrix} 1 & 0 & 0 & -20 \end{bmatrix} \\ R_2^{**} \begin{bmatrix} 0 & 1 & 0 & 8 \end{bmatrix} \\ R_3^{***} \begin{bmatrix} 0 & 0 & 1 & 2 \end{bmatrix} \end{array}$$

This final matrix is in reduced row echelon form.

19. The given linear system written in augmented matrix form is

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \left( \begin{array}{cccc|c} 1 & 2 & -3 & -5 & -13 \\ 3 & 1 & 4 & -4 & 5 \\ 2 & -2 & 3 & -10 & 7 \\ 1 & 1 & 2 & 2 & 0 \end{array} \right)$$

Interchange top and bottom rows because it is easier to deal with the numbers on the bottom row:

$$\begin{array}{l} R_1^* = R_4 \\ R_2 \\ R_3 \\ R_4^* = R_1 \end{array} \left( \begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 3 & 1 & 4 & -4 & 5 \\ 2 & -2 & 3 & -10 & 7 \\ 1 & 2 & -3 & -5 & -13 \end{array} \right)$$

Carrying out the row operations  $R_2 - 3R_1^*$ ,  $R_3 - 2R_1^*$  and  $R_4^* - R_1^*$ :

$$\begin{array}{l} R_1^* \\ R_2^* = R_2 - 3R_1^* \\ R_3^* = R_3 - 2R_1^* \\ R_4^{**} = R_4^* - R_1^* \end{array} \left( \begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 0 & -2 & -2 & -10 & 5 \\ 0 & -4 & -1 & -14 & 7 \\ 0 & 1 & -5 & -7 & -13 \end{array} \right)$$

Execute the row operation  $R_3^* - 2R_2^*$ :

$$\begin{array}{l} R_1^* \\ R_2^* \\ R_3^{**} = R_3^* - 2R_2^* \\ R_4^{**} \end{array} \left( \begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 0 & -2 & -2 & -10 & 5 \\ 0 & 0 & 3 & 6 & -3 \\ 0 & 1 & -5 & -7 & -13 \end{array} \right)$$

Carrying out the row operation  $2R_4^{**} + R_2^*$ :

$$\begin{array}{l} R_1^* \\ R_2^* \\ R_3^{**} \\ R_4^\dagger = 2R_4^{**} + R_2^* \end{array} \left( \begin{array}{cccc|c} 1 & 1 & 2 & 2 & 0 \\ 0 & -2 & -2 & -10 & 5 \\ 0 & 0 & 3 & 6 & -3 \\ 0 & 0 & -12 & -24 & -21 \end{array} \right)$$

Executing  $R_4^\dagger + 4R_3^{**}$ :

$$\begin{array}{l} R_1^* \\ R_2^* \\ R_3^{**} \\ R_4^{\dagger\dagger} = R_4^\dagger + 4R_3^{**} \end{array} \left( \begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ 1 & 1 & 2 & 2 & 0 \\ 0 & -2 & -2 & -10 & 5 \\ 0 & 0 & 3 & 6 & -3 \\ 0 & 0 & 0 & 0 & -33 \end{array} \right)$$

From the last row we have

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = -33 \text{ which means that } 0 = -33$$

This is impossible so the linear system is inconsistent.

20. Let  $\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$  then we have

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \quad \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Expanding each of these out

$$\begin{aligned} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} &= \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} && \begin{matrix} b+c=-1 \\ e+f=0 \\ h+i=2 \end{matrix} \\ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} && \begin{matrix} a+c=0 \\ d+f=-1 \\ g+i=2 \end{matrix} \\ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} && \begin{matrix} a+b=1 \\ d+e=1 \\ g+h=2 \end{matrix} \end{aligned}$$

Solving these gives

$$a=1, b=0, c=-1, d=0, e=1, f=-1, g=1, h=1 \text{ and } i=1$$

Putting these values into the matrix  $\mathbf{A}$  and checking:

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}, \quad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Thus  $\mathbf{A} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$ .

21. We need to find the reduced row echelon form, rref, of

$$\begin{matrix} R_1 \\ R_2 \\ R_3 \\ R_4 \end{matrix} \begin{pmatrix} 2 & 2 & 0 & 2 \\ -1 & -1 & 2 & 1 \\ 2 & 2 & -1 & 1 \\ -1 & -1 & 1 & 0 \end{pmatrix}$$

Interchanging rows and multiplying by  $-1$  gives:

$$\begin{matrix} R_1' = -R_4 \\ R_2' = R_1 \\ R_3 \\ R_4' = -R_2 \end{matrix} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 2 & 2 & 0 & 2 \\ 2 & 2 & -1 & 1 \\ 1 & 1 & -2 & -1 \end{pmatrix}$$

Executing the row operations  $R_2' - 2R_1'$ ,  $R_3 - 2R_1'$  and  $R_4' - R_1'$ :

$$\begin{array}{l} R_1' \\ R_2'' = R_2' - 2R_1' \\ R_3' = R_3 - 2R_1' \\ R_4'' = R_4' - R_1' \end{array} \left( \begin{array}{cccc} 1 & 1 & -1 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{array} \right)$$

Carrying out the row operations  $R_2''/2$  and  $-R_4''$  gives

$$\begin{array}{l} R_1' \\ R_2''' = R_2''/2 \\ R_3' = R_3 - 2R_1' \\ R_4''' = -R_4'' \end{array} \left( \begin{array}{cccc} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array} \right)$$

Executing the row operations  $R_3' - R_2'''$  and  $R_4''' - R_2'''$  gives:

$$\begin{array}{l} R_1' \\ R_2''' = R_2''/2 \\ R_3^* = R_3' - R_2''' \\ R_4^* = R_4''' - R_2''' \end{array} \left( \begin{array}{cccc} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Executing the row operation  $R_1' + R_2'''$  gives

$$\begin{array}{l} R_1' + R_2''' \\ R_2''' = R_2''/2 \\ R_3^* \\ R_4^* \end{array} \left( \begin{array}{cccc} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

This is now in reduced row echelon form (rref).

22. Writing the augmented matrix for the given linear system:

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array} \left( \begin{array}{ccccc|c} 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 1 & -3 & 6 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{array} \right)$$

Interchanging rows 1 and 2 gives

$$\begin{array}{l} R_1^* = R_2 \\ R_2^* = R_1 \\ R_3 \\ R_4 \end{array} \left( \begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 2 & 3 & 1 & 4 & -9 & 17 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{array} \right)$$

Carrying out the row operations  $R_2^* - 2R_1^*$ ,  $R_3 - R_1^*$  and  $R_4 - 2R_1^*$ :

$$\begin{array}{l} R_1^* \\ R_2^{**} = R_2^* - 2R_1^* \\ R_3^* = R_3 - R_1^* \\ R_4^* = R_4 - 2R_1^* \end{array} \left( \begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 & -2 & 2 \end{array} \right)$$

Subtracting the bottom two rows,  $R_4^* - R_3^*$ :

$$\begin{array}{l} R_1^* \\ R_2^{**} \\ R_3^* \\ R_4^{**} = R_4^* - R_3^* \end{array} \left( \begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

Executing the row operation  $R_2^{**} - 2R_3^*$ :

$$\begin{array}{l} R_1^* \\ R_2^{***} = R_2^{**} - 2R_3^* \\ R_3^* \\ R_4^{**} \end{array} \left( \begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

Carrying out the row operation  $R_1^* - R_3^*$  gives

$$\begin{array}{l} R_1^{**} = R_1^* - R_3^* \\ R_2^{***} \\ R_3^* \\ R_4^{**} \end{array} \left( \begin{array}{ccccc|c} 1 & 1 & 1 & 0 & -1 & 4 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right)$$

Carrying out the row operation  $R_1^{**} - R_2^{***}$ :

$$\begin{array}{l} R_1^{**} - R_2^{***} \\ R_2^{***} \\ R_3^* \\ R_4^{**} \end{array} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ \left( \begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -2 & 3 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \end{array}$$

This is now in reduced row echelon form. We have 5 unknowns and only 3 non-zero equations so there are  $5 - 3 = 2$  free variables. Which of the above are free variables?  $x_3$  and  $x_5$ . Let  $x_3 = s$  and  $x_5 = t$ . From the third row  $R_3^*$  we have

$$x_4 - 2x_5 = 2 \Rightarrow x_4 = 2x_5 + 2 = 2t + 2$$

From the second row  $R_2^{***}$  we have

$$x_2 - x_3 + x_5 = 1 \Rightarrow x_2 = 1 + x_3 - x_5 = 1 + s - t$$

From the top row we have

$$x_1 + 2x_3 - 2x_5 = 3 \Rightarrow x_1 = 3 + 2x_3 - 2x_5 = 3 + 2t - 2s$$

Hence our solution in vector form is given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 + 2t - 2s \\ 1 + s - t \\ s \\ 2t + 2 \\ t \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

23. (a) We need to find the inverse of the given matrix **A** by applying elementary row operations on  $[\mathbf{A} \mid \mathbf{I}]$ . Need to convert (if possible) this to  $[\mathbf{I} \mid \mathbf{B}]$  then **B** is the inverse of matrix **A**.



$$[\mathbf{A} \mid \mathbf{I}] = \begin{array}{c} R_1 \\ R_2 \end{array} \left[ \begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ -5 & 2 & 0 & 1 \end{array} \right]$$

Carrying out the row operation  $R_2 + 5R_1$  gives

$$\begin{array}{c} R_1 \\ R_2^* = R_2 + 5R_1 \end{array} \left[ \begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 2 & 5 & 1 \end{array} \right]$$

Divide the bottom row by 2, that is  $R_2^*/2$  gives

$$\begin{array}{c} R_1 \\ R_2^*/2 \end{array} \left[ \begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 5/2 & 1/2 \end{array} \right]$$

Hence the inverse of the matrix  $\mathbf{A}$  is  $\mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 \\ 5/2 & 1/2 \end{bmatrix}$ . The elementary matrices are given

by examining the stated row operations above. The matrix  $\mathbf{E}_1$  is given by  $R_2 + 5R_1$  which is

$$\mathbf{E}_1 = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$$

The elementary matrix  $\mathbf{E}_2$  is defined by the above row operation  $R_2/2$ :

$$\mathbf{E}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Hence the matrix  $\mathbf{A}^{-1} = \mathbf{E}_2\mathbf{E}_1$ . You may like to check that this is indeed the case:

$$\mathbf{E}_2\mathbf{E}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 5/2 & 1/2 \end{pmatrix} = \mathbf{A}^{-1}$$

(b) *How can we write the matrix  $\mathbf{A}$  as a product of elementary matrices?*

From part (a) we have  $\mathbf{A}^{-1} = \mathbf{E}_2\mathbf{E}_1$ . Therefore

$$\mathbf{A} = (\mathbf{A}^{-1})^{-1} = (\mathbf{E}_2\mathbf{E}_1)^{-1} = \mathbf{E}_1^{-1}\mathbf{E}_2^{-1}$$

$\mathbf{E}_1 = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$  therefore  $\mathbf{E}_1^{-1} = \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix}$ . Similarly

$$\mathbf{E}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \text{ gives } \mathbf{E}_2^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Hence  $\mathbf{E}_1^{-1}\mathbf{E}_2^{-1} = \begin{pmatrix} 1 & 0 \\ -5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -5 & 2 \end{pmatrix} = \mathbf{A}$ .

24. *What do you notice about the given matrices?*

$$\mathbf{A} = \begin{bmatrix} 5 & 1 & 9 \\ 2 & 4 & 0 \\ 1 & 1 & 0 \\ -1 & 3 & -2 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 4 & 0 \\ 5 & 1 & 9 \\ -1 & 3 & -2 \end{bmatrix}$$

Matrix  $\mathbf{B}$  is matrix  $\mathbf{A}$  with the first and third rows swapped over. *How do we write this as an elementary matrix?*

The elementary matrix is the 4 by 4 identity matrix with a single row operation of swapping the first and third rows which is given by:

$$\mathbf{E} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

25. We are given

$$\left( \mathbf{A}^T - 3 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

*How do we find the matrix  $\mathbf{A}$ ?*

First by taking the inverse of both sides:

$$\left( \mathbf{A}^T - 3 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \quad \left[ \text{Because } (\mathbf{X}^{-1})^{-1} = \mathbf{X} \right]$$

$$\mathbf{A}^T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} + 3 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

In order to find  $\mathbf{A}$  we need to take the transpose of both sides because  $(\mathbf{A}^T)^T = \mathbf{A}$ .

We have

$$\begin{aligned} \mathbf{A}^T &= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} + 3 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} + 3 \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ -3 & 9 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -4 & 11 \end{bmatrix} \end{aligned}$$

Therefore  $\mathbf{A} = (\mathbf{A}^T)^T = \begin{bmatrix} 4 & 5 \\ -4 & 11 \end{bmatrix}^T = \begin{bmatrix} 4 & -4 \\ 5 & 11 \end{bmatrix}$ . This is option ④.

26. *How can we find the matrix  $\mathbf{X}$  from the given formula*

$$\mathbf{B}(\mathbf{X} + \mathbf{C}) = \mathbf{D} ?$$

By transposing this formula:

$$(\mathbf{X} + \mathbf{C}) = \mathbf{B}^{-1}\mathbf{D}$$

$$\mathbf{X} = \mathbf{B}^{-1}\mathbf{D} - \mathbf{C}$$

We need to find the inverse matrix  $\mathbf{B}^{-1}$ . *How?*

By converting the augmented matrix  $[\mathbf{B} \mid \mathbf{I}]$  to  $[\mathbf{I} \mid \mathbf{A}]$ . The matrix  $\mathbf{A}$  is the inverse of  $\mathbf{B}$ .

We have  $[\mathbf{B} \mid \mathbf{I}]$  is equal to

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \left[ \begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 3 & 6 & 0 & 0 & 1 \end{array} \right]$$

Carrying out the row operation  $R_3 - R_1$  gives:

$$\begin{array}{l} R_1 \\ R_2 \\ R_3' = R_3 - R_1 \end{array} \left[ \begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array} \right]$$

Carrying out the row operation  $R_2 - 2R_3'$ :

$$\begin{array}{l} R_1 \\ R_2' = R_2 - 2R_3' \\ R_3' \end{array} \left[ \begin{array}{ccc|ccc} 1 & 3 & 5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array} \right]$$

Executing the row operation  $R_1 - 3R_2'$ :

$$\begin{array}{l} R_1' = R_1 - 3R_2' \\ R_2' \\ R_3' \end{array} \left[ \begin{array}{ccc|ccc} 1 & 0 & 5 & -5 & -3 & 6 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array} \right]$$

The last row operation is  $R_1' - 5R_3'$ :

$$\begin{array}{l} R_1'' = R_1' - 5R_3' \\ R_2' \\ R_3' \end{array} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -3 & 1 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array} \right]$$

Thus  $\mathbf{B}^{-1} = \begin{bmatrix} 0 & -3 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$ . Substituting this  $\mathbf{B}^{-1} = \begin{bmatrix} 0 & -3 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$  and

$\mathbf{C} = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 5 & 0 & -2 \end{bmatrix}$ ,  $\mathbf{D} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & -2 & 0 \end{bmatrix}$  into  $\mathbf{X} = \mathbf{B}^{-1}\mathbf{D} - \mathbf{C}$  gives

$$\begin{aligned} \mathbf{X} &= \begin{bmatrix} 0 & -3 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & -2 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 5 & 0 & -2 \end{bmatrix} \\ &= \begin{bmatrix} -3 & -2 & 3 \\ -5 & 6 & 3 \\ 3 & -3 & -2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 5 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -6 & -2 & 2 \\ -5 & 4 & 3 \\ -2 & -3 & 0 \end{bmatrix} \end{aligned}$$

27. a. The augmented matrix for the given linear system is

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \left( \begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ -1 & 3 & 2 & -2 \\ 2 & 1 & 1 & 1 \end{array} \right)$$

Executing the row operations  $R_2 + R_1$  and  $R_3 - 2R_1$ :

$$\begin{array}{l} R_1 \\ R_2' = R_2 + R_1 \\ R_3' = R_3 - 2R_1 \end{array} \left( \begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 4 & 3 & 3 \\ 0 & -1 & -1 & -9 \end{array} \right)$$

Carrying out the row operation  $4R_3' + R_2'$ :

$$\begin{array}{l} R_1 \\ R_2' \\ R_3'' = 4R_3' + R_2' \end{array} \left( \begin{array}{ccc|c} x & y & z & \\ 1 & 1 & 1 & 5 \\ 0 & 4 & 3 & 3 \\ 0 & 0 & -1 & -33 \end{array} \right)$$

From the bottom row we have  $z = 33$ . Substituting this into the expansion of the middle row

$$4y + 3z = 3 \Rightarrow 4y + 99 = 3 \Rightarrow y = -24$$

Substituting these two results  $y = -24$  and  $z = 33$  into the expansion of the first row gives

$$x + y + z = 5 \Rightarrow x - 24 + 33 = 5 \Rightarrow x = -4$$

Thus our solution to the given linear system is  $x = -4$ ,  $y = -24$  and  $z = 33$ .

b. Writing the given linear system in matrix form is

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 3 & 2 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$$

The inverse of the 3 by 3 matrix is (use row operations to find the inverse):

$$\begin{pmatrix} -1 & 0 & 1 \\ -5 & 1 & 3 \\ 7 & -1 & -4 \end{pmatrix}$$

Hence we have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ -5 & 1 & 3 \\ 7 & -1 & -4 \end{pmatrix} \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ -24 \\ 33 \end{pmatrix} \text{ gives } x = -4, y = -24 \text{ and } z = 33$$

c. The given system will have **no** solution if in the above row operations the  $-1$  in the bottom row is zero but the right hand side of the vertical bar is non-zero.

$$\begin{array}{l} R_1 \\ R_2' \\ R_3'' = 4R_3' + R_2' \end{array} \left( \begin{array}{ccc|c} x & y & z & \\ 1 & 1 & 1 & 5 \\ 0 & 4 & 3 & 3 \\ 0 & 0 & -1 & -33 \end{array} \right) \quad \begin{array}{l} \text{No solution} \\ \text{if this is} \\ \text{zero.} \end{array}$$

We use the row operations stated above in the opposite direction. In order to achieve a 0 in this position we need to nominate  $c_1$  the coefficient of the  $z$  variable in the operation before, that is

$$\begin{array}{l} R_1 \\ R_2' = R_2 + R_1 \\ R_3' = R_3 - 2R_1 \end{array} \left( \begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 4 & 3 & 3 \\ 0 & -1 & c_1 & -9 \end{array} \right)$$

The row operation was  $R_3'' = 4R_3' + R_2'$  so we have  $4c_1 + 3 = 0$  which gives  $c_1 = -3/4$ . In order to get  $c_1 = -3/4$  in this position we let  $c_2$  be the  $z$  coefficient in the previous row operation, that is

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \left( \begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ -1 & 3 & 2 & -2 \\ 2 & 1 & c_2 & 1 \end{array} \right)$$

The row operation was  $R_3' = R_3 - 2R_2$  so we have  $c_2 - 2 = -3/4$  which gives  $c_2 = 5/4$ . Thus our  $z$  coefficient is  $5/4$  in order for the linear system to be **inconsistent** which means that it has **no** solution.

28. We need to prove that  $\mathbf{A}^{-1}$  is symmetric provided  $\mathbf{A}$  is symmetric. *How do we prove this?*

Required to show that  $(\mathbf{A}^{-1})^T = \mathbf{A}^{-1}$

*Proof.* We have

$$\begin{aligned} (\mathbf{A}^{-1})^T &= (\mathbf{A}^T)^{-1} && \left[ \text{Because } (\mathbf{X}^{-1})^T = (\mathbf{X}^T)^{-1} \right] \\ &= \mathbf{A}^{-1} && \left[ \text{Because } \mathbf{A} \text{ is symmetric} \right] \end{aligned}$$

This is our required result.

29. We are given  $\mathbf{A}^2 - \mathbf{A} + \mathbf{I} = \mathbf{O}$  and need to show that the inverse of the matrix  $\mathbf{A}$  is given by  $\mathbf{I} - \mathbf{A}$ .

*Proof.*

We pre-multiply  $\mathbf{A}$  by  $\mathbf{I} - \mathbf{A}$ :

$$\begin{aligned} (\mathbf{I} - \mathbf{A})\mathbf{A} &= \mathbf{I}\mathbf{A} - \mathbf{A}^2 \\ &= \mathbf{A} - \mathbf{A}^2 = \mathbf{A} - (\mathbf{A} - \mathbf{I}) && \left[ \text{Because we are given } \mathbf{A}^2 - \mathbf{A} + \mathbf{I} = \mathbf{O} \right] \\ &= \mathbf{I} \end{aligned}$$

We have  $(\mathbf{I} - \mathbf{A})\mathbf{A} = \mathbf{I}$ . Next we post-multiply  $\mathbf{A}$  by  $\mathbf{I} - \mathbf{A}$ :

$$\mathbf{A}(\mathbf{I} - \mathbf{A}) = \mathbf{A}\mathbf{I} - \mathbf{A}^2 = \mathbf{A} - (\mathbf{A} - \mathbf{I}) = \mathbf{I}$$

Thus we have  $\mathbf{A}(\mathbf{I} - \mathbf{A}) = \mathbf{I}$ . Combining these results shows that the matrix  $\mathbf{I} - \mathbf{A}$  is the inverse of the matrix  $\mathbf{A}$ .

30. (a) A square matrix  $\mathbf{A}$  is invertible if and only if there exists a square matrix  $\mathbf{B}$  such that  $\mathbf{AB} = \mathbf{BA} = \mathbf{I}$  where  $\mathbf{I}$  is the identity matrix

Need to prove that this  $\mathbf{B}$  is unique.

*Proof.*

Suppose there exists a square matrix  $\mathbf{C}$  such that

$$\mathbf{AC} = \mathbf{CA} = \mathbf{I}$$

*What do we need to prove?*

Required to prove that  $\mathbf{B} = \mathbf{C}$ .

We have  $\mathbf{AB} = \mathbf{AC} = \mathbf{I}$ . Pre-multiply both sides by  $\mathbf{B}$ :

$$\begin{aligned} \mathbf{B}(\mathbf{AB}) &= \mathbf{B}(\mathbf{AC}) \\ (\mathbf{BA})\mathbf{B} &= (\mathbf{BA})\mathbf{C} \\ \mathbf{IB} &= \mathbf{IC} && \left[ \text{Because from above } \mathbf{BA} = \mathbf{I} \right] \\ \mathbf{B} &= \mathbf{C} \end{aligned}$$

Hence the inverse matrix  $\mathbf{B}$  is unique.

(b) We need to prove that the product of any finite number of invertible matrices is invertible.

*Proof.*

Let  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_n$  be a sequence of invertible matrices. Consider the product  $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\cdots\mathbf{A}_n$ . We use mathematical induction to show

$$(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\cdots\mathbf{A}_n)^{-1} = \mathbf{A}_n^{-1}\mathbf{A}_{n-1}^{-1}\cdots\mathbf{A}_2^{-1}\mathbf{A}_1^{-1}$$

Consider the result for  $n = 2$ :

$$(\mathbf{A}_1\mathbf{A}_2)^{-1} = \mathbf{A}_2^{-1}\mathbf{A}_1^{-1} \text{ because } (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Assume the result is true for  $n = k$ , that is

$$(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\cdots\mathbf{A}_k)^{-1} = \mathbf{A}_k^{-1}\mathbf{A}_{k-1}^{-1}\cdots\mathbf{A}_2^{-1}\mathbf{A}_1^{-1} \quad (*)$$

We apply these two to show the result for  $n = k + 1$ . *What do we need to prove?*

Required to prove

$$(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\cdots\mathbf{A}_k\mathbf{A}_{k+1})^{-1} = \mathbf{A}_{k+1}^{-1}\mathbf{A}_k^{-1}\cdots\mathbf{A}_2^{-1}\mathbf{A}_1^{-1}$$

Starting with the Left Hand Side we have

$$\begin{aligned} (\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\cdots\mathbf{A}_k\mathbf{A}_{k+1})^{-1} &= \mathbf{A}_{k+1}^{-1}(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\cdots\mathbf{A}_k)^{-1} && \left[ \text{Because } (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \right] \\ &= \mathbf{A}_{k+1}^{-1}\mathbf{A}_k^{-1}\cdots\mathbf{A}_2^{-1}\mathbf{A}_1^{-1} && \left[ \text{By } (*) \right] \end{aligned}$$

Hence we have our result. Therefore by mathematical induction we conclude that the product of a finite number of invertible matrices is invertible.

(c) The inverse of the given matrix is obtained by converting  $[\mathbf{A} \mid \mathbf{I}]$  to  $[\mathbf{I} \mid \mathbf{B}]$  and  $\mathbf{B}$  would be the inverse matrix of  $\mathbf{A}$ :

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array} \right]$$

Interchanging middle and bottom rows,  $R_2$  and  $R_3$ , gives

$$\begin{array}{l} R_1 \\ R_2' = R_3 \\ R_3' = R_2 \end{array} \left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 & 1 & 0 \end{array} \right]$$

Executing the row operation  $R_3' + R_1$  gives

$$\begin{array}{l} R_1 \\ R_2' \\ R_3^* = R_3' + R_1 \end{array} \left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 & 1 & 0 \end{array} \right]$$

*What are we trying to achieve?*

The identity matrix on the Left Hand Side. *How?*

Carry out the row operation  $R_3^* - R_2'$ :

$$\begin{array}{l} R_1 \\ R_2' \\ R_3^{**} = R_3^* - R_2' \end{array} \left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & -1 \end{array} \right]$$

Carrying out the row operation  $2R_1 - R_3^{**}$ :

$$\begin{array}{l} R_1' = 2R_1 - R_3^{**} \\ R_2' \\ R_3^{**} \end{array} \left[ \begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & -1 \end{array} \right]$$

Divide the top and bottom row by 2:

$$\begin{array}{l} R_1'/2 \\ R_2' \\ R_3^{**}/2 \end{array} \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1/2 \end{array} \right]$$

The inverse of the matrix  $\mathbf{A}$  is the matrix on the Right Hand Side:

$$\mathbf{A}^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & -1/2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 2 \\ 1 & 1 & -1 \end{bmatrix}$$

You may check that  $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$ .

31. We are given  $\mathbf{A}\mathbf{X}\mathbf{A}^{-1} = \mathbf{B}$  and  $\mathbf{A}$ ,  $\mathbf{B}$  are invertible (non-singular) matrices. We have

$$\begin{aligned} \mathbf{A}\mathbf{X}\mathbf{A}^{-1} &= \mathbf{B} \\ \mathbf{A}^{-1}(\mathbf{A}\mathbf{X}\mathbf{A}^{-1}) &= \mathbf{A}^{-1}\mathbf{B} && \text{[Pre multiplying both sides by } \mathbf{A}^{-1}] \\ \underbrace{(\mathbf{A}^{-1}\mathbf{A})}_{=\mathbf{I}}\mathbf{X}\mathbf{A}^{-1} &= \mathbf{A}^{-1}\mathbf{B} \\ \mathbf{X}\mathbf{A}^{-1} &= \mathbf{A}^{-1}\mathbf{B} \\ (\mathbf{X}\mathbf{A}^{-1})\mathbf{A} &= \mathbf{A}^{-1}\mathbf{B}\mathbf{A} && \text{[Post multiplying both sides by } \mathbf{A}] \\ \mathbf{X}(\underbrace{\mathbf{A}^{-1}\mathbf{A}}_{=\mathbf{I}}) &= \mathbf{A}^{-1}\mathbf{B}\mathbf{A} \\ \mathbf{X} &= \mathbf{A}^{-1}\mathbf{B}\mathbf{A} \end{aligned}$$

Since  $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}\mathbf{A}$  and matrices  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{A}^{-1}$  are invertible therefore  $\mathbf{A}^{-1}\mathbf{B}\mathbf{A}$  and  $\mathbf{X}$  are invertible.

What is  $\mathbf{X}^{-1}$  equal to?

$$\begin{aligned} \mathbf{X}^{-1} &= (\mathbf{A}^{-1}\mathbf{B}\mathbf{A})^{-1} \\ &= (\mathbf{B}\mathbf{A})^{-1}(\mathbf{A}^{-1})^{-1} \\ &= \mathbf{A}^{-1}\mathbf{B}^{-1}\mathbf{A} && \left[ \text{Because } (\mathbf{B}\mathbf{A})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1} \text{ and } (\mathbf{A}^{-1})^{-1} = \mathbf{A} \right] \end{aligned}$$

Hence  $\mathbf{X}^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}\mathbf{A}$ .

32. We are given the linear system

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 3 & 0 & 5 \\ 0 & 1 & 1 & 0 & 2 & 1 & 10 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \\ 16 \end{bmatrix}$$

The augmented matrix is given by

$$\begin{array}{l} R_1 \left[ \begin{array}{ccccccc|c} 0 & 1 & 1 & 0 & 2 & 0 & 4 & 7 \end{array} \right] \\ R_2 \left[ \begin{array}{ccccccc|c} 0 & 0 & 0 & 1 & 3 & 0 & 5 & 8 \end{array} \right] \\ R_3 \left[ \begin{array}{ccccccc|c} 0 & 1 & 1 & 0 & 2 & 1 & 10 & 16 \end{array} \right] \end{array}$$

Carrying out the row operation  $R_3 - R_1$  gives

$$\begin{array}{l} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ R_1 \left[ \begin{array}{ccccccc|c} 0 & 1 & 1 & 0 & 2 & 0 & 4 & 7 \end{array} \right] \\ R_2 \left[ \begin{array}{ccccccc|c} 0 & 0 & 0 & 1 & 3 & 0 & 5 & 8 \end{array} \right] \\ R'_3 = R_3 - R_1 \left[ \begin{array}{ccccccc|c} 0 & 0 & 0 & 0 & 0 & 1 & 6 & 9 \end{array} \right] \end{array}$$

How many free variables do we have in this case?

$7 - 3 = 4$ . Which of these variables are free?

$a, c, e$  and  $g$  because corresponding columns do **not** contain a leading 1. Let

$a = p, c = q, e = r$  and  $g = t$ . From the bottom row  $R'_3$  we have

$$f + 6g = 9 \quad \text{which gives } f = 9 - 6g = 9 - 6t$$

From the middle row  $R_2$  we have

$$d + 3e + 5g = 8 \quad \text{which gives } d = 8 - 3e - 5g = 8 - 3r - 5t$$

From the top row  $R_1$  we have

$$b + c + 2e + 4g = 7 \quad \text{implies that } b = 7 - c - 2e - 4g = 7 - q - 2r - 4t$$

Thus our solution set is given by

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \end{bmatrix} = \begin{bmatrix} p \\ 7 - q - 2r - 4t \\ q \\ 8 - 3r - 5t \\ r \\ 9 - 6t \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ 0 \\ 8 \\ 0 \\ 9 \\ 0 \end{bmatrix} + p \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + q \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} 0 \\ -2 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -4 \\ 0 \\ -5 \\ 0 \\ -6 \\ 1 \end{bmatrix}$$

33. We need to prove that  $\mathbf{AB} = \mathbf{I} \Leftrightarrow \mathbf{BA} = \mathbf{I}$ .

*Proof.* ( $\Rightarrow$ ). Assume that  $\mathbf{AB} = \mathbf{I}$ . This means that matrix  $\mathbf{B}$  is the right inverse of the matrix  $\mathbf{A}$ . Using this

$$\begin{aligned} \mathbf{B} &= \mathbf{BI} \\ &= \mathbf{B}(\mathbf{AB}) \\ &= (\mathbf{BA})\mathbf{B} \end{aligned}$$



Post-multiply this  $\mathbf{B} = (\mathbf{BA})\mathbf{B}$  by  $\mathbf{B}^{-1}$ :

$$\begin{aligned}\mathbf{BB}^{-1} &= (\mathbf{BA})\mathbf{BB}^{-1} \\ \mathbf{I} &= \mathbf{BAI} = \mathbf{BA}\end{aligned}$$

We have  $\mathbf{BA} = \mathbf{I}$ .

( $\Leftarrow$ ). Similarly we go the other way, that is we assume  $\mathbf{BA} = \mathbf{I}$  and deduce  $\mathbf{AB} = \mathbf{I}$ .

$$\begin{aligned}\mathbf{A} &= \mathbf{AI} \\ &= \mathbf{A}(\mathbf{BA}) \\ &= (\mathbf{AB})\mathbf{A}\end{aligned}$$

Post-multiply this  $\mathbf{A} = (\mathbf{AB})\mathbf{A}$  by  $\mathbf{A}^{-1}$ :

$$\begin{aligned}\mathbf{AA}^{-1} &= (\mathbf{AB})\mathbf{AA}^{-1} \\ \mathbf{I} &= \mathbf{ABI} = \mathbf{AB}\end{aligned}$$

We have  $\mathbf{AB} = \mathbf{I}$ .

34. We need to prove if  $\mathbf{AB}$  are square matrices and  $\mathbf{AB}$  is invertible (non-singular) then both  $\mathbf{A}$  and  $\mathbf{B}$  are invertible with  $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ .

*What do we need to show?*

It is enough to prove that  $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{I}$  because by result of question 33 we have  $(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{I}$ .

*Proof.*

Proving the first result:

$$\begin{aligned}(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) &= \mathbf{A}(\underbrace{\mathbf{BB}^{-1}}_{=\mathbf{I}})\mathbf{A}^{-1} \\ &= \mathbf{AIA}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}\end{aligned}$$

By the result of question 33  $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{I} \Leftrightarrow (\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{I}$

Thus  $\mathbf{B}^{-1}\mathbf{A}^{-1}$  is the inverse of the matrix multiplication  $\mathbf{AB}$ . Hence

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

We have proven our required result.

35. (a) We are given the diagonal matrix  $\mathbf{D} = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & 0 & a_{nn} \end{pmatrix}$  and need to prove that

$$\mathbf{D}^p = \begin{pmatrix} a_{11}^p & 0 & \cdots & 0 \\ 0 & a_{22}^p & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & 0 & a_{nn}^p \end{pmatrix}$$

*How?*

Using mathematical induction. Clearly the result is true for  $p = 1$ . Assume the result is true for  $p = k$ :

$$\mathbf{D}^k = \begin{pmatrix} a_{11}^k & 0 & \cdots & 0 \\ 0 & a_{22}^k & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & 0 & a_{nn}^k \end{pmatrix}$$

Required to prove the result for  $p = k + 1$ . We have

$$\begin{aligned} \mathbf{D}^{k+1} = \mathbf{D}^k \mathbf{D} &= \begin{pmatrix} a_{11}^k & 0 & \cdots & 0 \\ 0 & a_{22}^k & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & 0 & a_{nn}^k \end{pmatrix} \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & 0 & a_{nn} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}^k a_{11} & 0 & \cdots & 0 \\ 0 & a_{22}^k a_{22} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & 0 & a_{nn}^k a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11}^{k+1} & 0 & \cdots & 0 \\ 0 & a_{22}^{k+1} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & 0 & a_{nn}^{k+1} \end{pmatrix} \end{aligned}$$

Hence by mathematical induction we have our result.

(b) Multiplying out the matrices  $\mathbf{D}$  and  $\mathbf{D}^{-1}$ :

$$\mathbf{D}\mathbf{D}^{-1} = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & 0 & a_{nn} \end{pmatrix} \begin{pmatrix} a_{11}^{-1} & 0 & \cdots & 0 \\ 0 & a_{22}^{-1} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & 0 & a_{nn}^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & 0 & 1 \end{pmatrix} = \mathbf{I}$$

We have  $\mathbf{D}\mathbf{D}^{-1} = \mathbf{I}$ . By the result of question 33 we have  $\mathbf{D}\mathbf{D}^{-1} = \mathbf{I} \Leftrightarrow \mathbf{D}^{-1}\mathbf{D} = \mathbf{I}$ . Hence the

inverse matrix is given by  $\mathbf{D}^{-1} = \begin{pmatrix} a_{11}^{-1} & 0 & \cdots & 0 \\ 0 & a_{22}^{-1} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \vdots & 0 & a_{nn}^{-1} \end{pmatrix}.$