

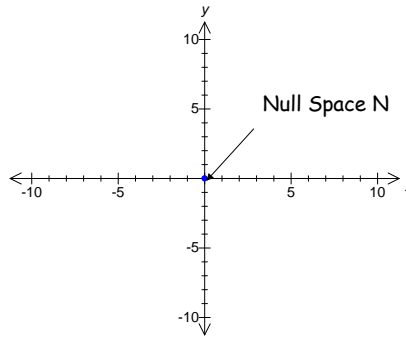
## Complete Solutions to Exercises 3.6

1. We need to find null space of each matrix.

(a) Let  $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  then the null space is the vector  $\mathbf{x}$  which satisfies  $\mathbf{Ax} = \mathbf{0}$ . We have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ gives } x = 0 \text{ and } y = 0$$

The graph of the null space is the zero vector (or the origin in the  $x$ - $y$  plane), that is



(b) Let  $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$  then the reduced row echelon form of this matrix is  $\mathbf{R} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ . We can find the null space of the given matrix  $\mathbf{B}$  by solving the equivalent system  $\mathbf{Rx} = \mathbf{0}$ :

$$\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Expanding the top row we have

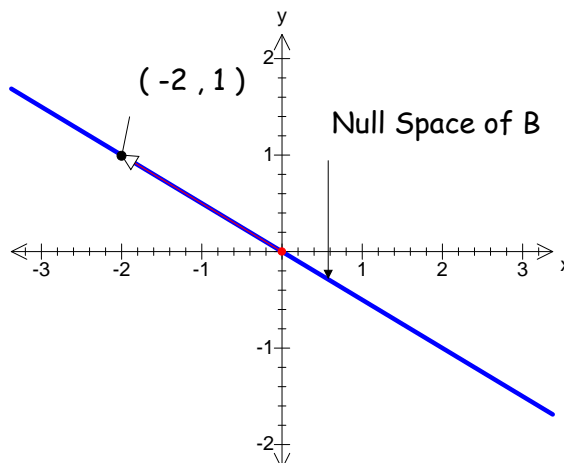
$$x + 2y = 0 \text{ gives } x = -2y$$

Let  $y = s$  where  $s$  is any real number then we have  $x = -2y = -2s$  and the solution is

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2s \\ s \end{pmatrix} = s \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Hence null space  $N$  of the given matrix  $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$  is  $N = \left\{ s \begin{pmatrix} -2 \\ 1 \end{pmatrix} \mid s \in \mathbb{R} \right\}$ . What does the null space  $N$  look like?

It is the straight line spanned by the vector  $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$  which is



(c) Similarly we have  $\mathbf{C} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \\ 6 & 10 \end{pmatrix}$  and the reduced row echelon form of this is  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ .

The null space is the vector  $\mathbf{x}$  which satisfies the equivalent system

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{gives} \quad x = 0 \quad \text{and} \quad y = 0$$

The null space  $N$  is the zero vector as shown in solution to part (a).

(d) Let  $\mathbf{D} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ . The reduced row echelon form of this matrix is  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ . As for the

solution of part (c) we have the null space  $N$  is given by the zero vector or the origin of the  $x$ - $y$  plane  $\{(0, 0)\}$  and is shown in part (a).

Parts (c) and (d) are both overdetermined systems.

2. (a) Writing out the given equations in matrix form we have  $\mathbf{Ax} = \mathbf{O}$  where

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & -3 \\ 4 & -5 & -6 \\ 7 & -8 & -9 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \mathbf{O} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Converting the matrix  $\mathbf{A}$  into reduced row echelon form by using MATLAB we have

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & -3 \\ 4 & -5 & -6 \\ 7 & -8 & -9 \end{pmatrix} \quad \mathbf{R} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

By examining the equivalent system  $\mathbf{Rx} = \mathbf{O}$  we have

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Expanding the middle row we have  $y + 2z = 0$  which gives  $y = -2z$ . Let  $z = s$  where  $s$  is any real number then we have  $y = -2z = -2s$ .

Expanding the top row we have

$$x + z = 0 \quad \text{gives} \quad x = -z = -s$$

Thus  $x = -s$ ,  $y = -2s$  and  $z = s$ . Our general solution is

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -s \\ -2s \\ s \end{pmatrix} = s \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} \quad \text{where } s \text{ is in}$$

(b) Similarly writing out the given equations in matrix form as  $\mathbf{Bx} = \mathbf{O}$  where

$$\mathbf{B} = \begin{pmatrix} 2 & -2 & -2 \\ 4 & -4 & -4 \\ 8 & -8 & -8 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \mathbf{O} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Putting matrix  $\mathbf{B}$  into reduced row echelon form gives

$$\mathbf{B} = \begin{pmatrix} 2 & -2 & -2 \\ 4 & -4 & -4 \\ 8 & -8 & -8 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{R}$$

Solving the system  $\mathbf{R}\mathbf{x} = \mathbf{O}$  we have

$$\begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Expanding the top row yields

$$x - y - z = 0 \text{ which gives } x = y + z$$

Let  $y = s$  and  $z = t$  then  $x = s + t$  and we have the general solution

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s+t \\ s \\ t \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ where } s, t \in \mathbb{R}$$

(c) Repeating the same process as above, the given linear equations can be written in matrix form as  $\mathbf{C}\mathbf{x} = \mathbf{O}$  where

$$\mathbf{C} = \begin{pmatrix} 2 & 9 & -3 \\ 5 & 6 & -1 \\ 9 & 8 & -9 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } \mathbf{O} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Converting matrix  $\mathbf{C}$  into reduced row echelon form matrix  $\mathbf{R}$  gives

$$\mathbf{C} = \begin{pmatrix} 2 & 9 & -3 \\ 5 & 6 & -1 \\ 9 & 8 & -9 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{R}$$

Matrix  $\mathbf{C}$  is of full rank, so  $\mathbf{C}\mathbf{x} = \mathbf{O} \Rightarrow \mathbf{x} = \mathbf{O}$  is the only solution:

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(d) The given linear equations in matrix form can be written as  $\mathbf{D}\mathbf{x} = \mathbf{O}$  where

$$\mathbf{D} = \begin{pmatrix} -3 & 1 & -1 \\ 2 & 5 & -7 \\ 4 & 8 & -4 \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } \mathbf{O} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The reduced row echelon form of matrix  $\mathbf{D}$  is given by

$$\mathbf{D} = \begin{pmatrix} -3 & 1 & -1 \\ 2 & 5 & -7 \\ 4 & 8 & -4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{R}$$

As with part (c) the solution is  $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$

3. Notice all of these matrices are the coefficient matrices of question 2 and the null space is the solution space  $\mathbf{x}$  evaluated above.

(a) By solution to question 2(a) above we have

$$\mathbf{x} = s \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = s\mathbf{u} \quad \text{where } \mathbf{u} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

Thus the null space of the given matrix  $\mathbf{A}$  is  $\{s\mathbf{u} \mid s \in \mathbb{R}\}$  where  $\mathbf{u}$  is the vector above.

*What is nullity of the given matrix  $\mathbf{A}$  equal to?*

We only have one vector in the null space of  $\mathbf{A}$  therefore this vector  $\mathbf{u}$  is a basis for the null space which means that  $\text{nullity}(\mathbf{A}) = 1$ .

*What is  $\text{rank}(\mathbf{A})$  equal to?*

The reduced row echelon form of matrix  $\mathbf{A}$  has 2 non-zero rows therefore  $\text{rank}(\mathbf{A}) = 2$ .

(b) By solution to question 2(b) above we have solution space is given by

$$\mathbf{x} = s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = s\mathbf{u} + t\mathbf{v} \quad \text{where } \mathbf{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad \text{and } \mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

The null space of matrix  $\mathbf{B}$  is  $N = \{s\mathbf{u} + t\mathbf{v} \mid s \in \mathbb{R}, t \in \mathbb{R}\}$ .

The vectors  $\mathbf{u}$  and  $\mathbf{v}$  are linearly independent and span the solution space therefore they are a basis (axes) for the null space  $N$ . Since the basis for the null space has 2 vectors,  $\mathbf{u}$  and  $\mathbf{v}$ , therefore  $\text{nullity}(\mathbf{B}) = 2$ .

*What is the rank of matrix  $\mathbf{B}$  equal to?*

The reduced row echelon form of matrix  $\mathbf{B}$  has only one non-zero row therefore  $\text{rank}(\mathbf{B}) = 1$ .

(c) Similarly for matrix  $\mathbf{C}$  the solution space is

$$\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Thus the null space of matrix  $\mathbf{C}$  is  $N = \{\mathbf{0}\}$  that is null space is just the zero vector.

*What is the nullity of matrix  $\mathbf{C}$  equal to?*

Remember from the definition of dimension we have that the dimension of the zero vector space is zero, so we have

$$\text{nullity}(\mathbf{C}) = 0$$

Since the reduced row echelon form of matrix  $\mathbf{C}$  is the 3 by 3 identity matrix therefore the number of non-zero rows is 3 which means  $\text{rank}(\mathbf{C}) = 3$ . Matrix  $\mathbf{C}$  is of full rank.

(d) This is very similar to part (c) because matrix  $\mathbf{D}$  is of full rank. We have

$$N = \{\mathbf{0}\}, \quad \text{nullity}(\mathbf{D}) = 0 \quad \text{and} \quad \text{rank}(\mathbf{D}) = 3$$

4. We have the same matrix as Example 36 of the main text. The reduced row echelon form matrix  $\mathbf{R}$  was also evaluated in that example and null space is the solution space  $\mathbf{x}$  of the equivalent system  $\mathbf{R}\mathbf{x} = \mathbf{0}$  which is

$$\begin{pmatrix} 1 & 0 & -1 & -2 & -3 & -4 & -5 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Expanding the second row we have

$$x_2 + 2x_3 + 3x_4 + 4x_5 + 5x_6 + 6x_7 = 0$$

which gives

$$x_2 = -2x_3 - 3x_4 - 4x_5 - 5x_6 - 6x_7 \quad (*)$$

Note that  $x_3, x_4, x_5, x_6$  and  $x_7$  are the free variables, so let

$$x_3 = s, \quad x_4 = t, \quad x_5 = p, \quad x_6 = q \quad \text{and} \quad x_7 = r$$

Substituting these into (\*) gives

$$\begin{aligned} x_2 &= -2x_3 - 3x_4 - 4x_5 - 5x_6 - 6x_7 \\ &= -2s - 3t - 4p - 5q - 6r \end{aligned}$$

Expanding the top row in the above matrix representation we have

$$x_1 - x_3 - 2x_4 - 3x_5 - 4x_6 - 5x_7 = 0$$

$$\begin{aligned} x_1 &= x_3 + 2x_4 + 3x_5 + 4x_6 + 5x_7 \\ &= s + 2t + 3p + 4q + 5r \end{aligned}$$

We have  $x_1 = s + 2t + 3p + 4q + 5r$ ,  $x_2 = -2s - 3t - 4p - 5q - 6r$ ,  $x_3 = s$ ,  $x_4 = t$ ,  $x_5 = p$ ,  $x_6 = q$  and  $x_7 = r$ :

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} s + 2t + 3p + 4q + 5r \\ -2s - 3t - 4p - 5q - 6r \\ s \\ t \\ p \\ q \\ r \end{pmatrix} = s\mathbf{v}_1 + t\mathbf{v}_2 + p\mathbf{v}_3 + q\mathbf{v}_4 + r\mathbf{v}_5$$

where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 4 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_5 = \begin{pmatrix} 5 \\ -6 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Our null space for the given matrix  $\mathbf{A}$  is

$$N = \left\{ s\mathbf{v}_1 + t\mathbf{v}_2 + p\mathbf{v}_3 + q\mathbf{v}_4 + r\mathbf{v}_5 \mid s \in \mathbb{R}, t \in \mathbb{R}, p \in \mathbb{R}, q \in \mathbb{R}, r \in \mathbb{R} \right\}$$

where the  $\mathbf{v}$  vectors are as shown above.

5. In each case we need to place the given matrix into (reduced) row echelon form.

(a) For the given matrix  $\mathbf{A}$  we have

$$\mathbf{A} = \begin{pmatrix} 1 & -4 & -9 \\ 2 & 5 & -7 \end{pmatrix} \quad \begin{pmatrix} \boxed{1} & 0 & -73/13 \\ 0 & \boxed{1} & 11/13 \end{pmatrix} = \mathbf{R}$$

Multiplying this by 13 gives us a simpler basis for the row space:

$$\left\{ \begin{pmatrix} 13 \\ 0 \\ -73 \end{pmatrix}, \begin{pmatrix} 0 \\ 13 \\ 11 \end{pmatrix} \right\}$$

*How do we find a basis for the column space of matrix  $\mathbf{A}$ ?*

The first two columns of matrix  $\mathbf{R}$  contain leading ones so a basis is the first 2 columns of

matrix  $\mathbf{A}$ . Hence  $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \end{pmatrix} \right\}$  and by applying row operations on these we obtain

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

We have  $\text{rank}(\mathbf{A}) = 2$  because there are 2 vectors in the basis for the column (or row) space of matrix  $\mathbf{A}$ .

*What is a basis for the null space of the given matrix  $\mathbf{A}$ ?*

We can solve the equivalent system  $\mathbf{R}\mathbf{x} = \mathbf{0}$ :

$$\begin{pmatrix} 1 & 0 & -73/13 \\ 0 & 1 & 11/13 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Expanding out the top row gives

$$x - \frac{73}{13}z = 0 \Rightarrow x = \frac{73}{13}z$$

Similarly from the bottom row we have  $y = -\frac{11}{13}z$ . Let  $z = s$  where  $s$  is any real number.

Substituting this into the above we have  $x = \frac{73}{13}z = \frac{73}{13}s$  and  $y = -\frac{11}{13}z = -\frac{11}{13}s$ .

Multiplying by 13 gives us

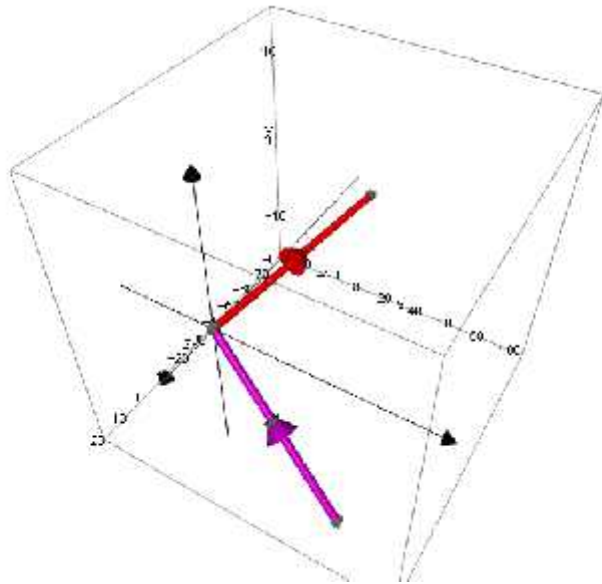
$$\mathbf{x} = s \begin{pmatrix} 73 \\ -11 \\ 13 \end{pmatrix}$$

A basis for the null space is  $\left\{ \begin{pmatrix} 73 \\ -11 \\ 13 \end{pmatrix} \right\}$ . Hence  $\text{nullity}(\mathbf{A}) = 1$ .

Later on in this exercise in question 11 we show that every vector in the null space is orthogonal to every vector in the row space. In this example we have

$$\begin{pmatrix} 13 \\ 0 \\ -73 \end{pmatrix} \cdot \begin{pmatrix} 73 \\ -11 \\ 13 \end{pmatrix} = (13 \times 73) + 0 + (-73 \times 11) = 0 \quad \text{and} \quad \begin{pmatrix} 0 \\ 13 \\ 11 \end{pmatrix} \cdot \begin{pmatrix} 73 \\ -11 \\ 13 \end{pmatrix} = 0$$

This example demonstrates that a basis vectors for the null and row space are orthogonal (perpendicular) to each other.



(b) Putting matrix  $\mathbf{B}$  into reduced row echelon form gives

$$\mathbf{B} = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ -14 & -37 \end{pmatrix} \quad \begin{pmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \\ 0 & 0 \end{pmatrix} = \mathbf{R}$$

A basis for the row space of matrix  $\mathbf{B}$  is the non-zero rows of matrix  $\mathbf{R}$ , that is  $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ .

To find a basis for the column space of matrix  $\mathbf{B}$  we can transpose this matrix and then place it into reduced row echelon form:

$$\mathbf{B}^T = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ -14 & -37 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & -14 \\ 3 & 5 & -37 \end{pmatrix}$$

$$\mathbf{B}^T = \begin{pmatrix} 1 & 2 & -14 \\ 3 & 5 & -37 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & -5 \end{pmatrix}$$

Thus a basis for the column space of matrix  $\mathbf{B}$  is  $\left\{ \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -5 \end{pmatrix} \right\}$ . We have  $\text{rank}(\mathbf{B}) = 2$ .

*What else do we need to find?*

A basis for the null space of matrix  $\mathbf{B}$ . Considering the equivalent homogeneous system  $\mathbf{R}\mathbf{x} = \mathbf{0}$  we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We have the solutions  $x = y = 0$ . However the zero vector  $\{\mathbf{0}\}$  **cannot** be a basis for the null space of  $\mathbf{B}$ . The null space has **no** basis. Hence  $\text{nullity}(\mathbf{B}) = 0$ .

(c) Similarly we have

$$\mathbf{C} = \begin{pmatrix} 1 & 3 & -9 & 5 \\ 2 & 6 & 7 & 1 \\ 1 & 3 & -8 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{R}$$

A basis for the row space of matrix  $\mathbf{C}$  is

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

We need to transpose the given matrix  $\mathbf{C}$  in order to find a basis for the column space of  $\mathbf{C}$ :

$$\mathbf{C}^T = \begin{pmatrix} 1 & 3 & -9 & 5 \\ 2 & 6 & 7 & 1 \\ 1 & 3 & -8 & 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ -9 & 7 & -8 \\ 5 & 1 & 1 \end{pmatrix}$$

Placing  $\mathbf{C}^T$  into reduced row echelon form gives

$$\mathbf{C}^T = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ -9 & 7 & -8 \\ 5 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The non-zero rows of the matrix on the Right Hand Side forms a basis for the column space

of the given matrix  $\mathbf{C}$ . Hence  $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$  is a basis for the column space of  $\mathbf{C}$ . We

have  $\text{rank}(\mathbf{C}) = 3$ . Hence the column space of matrix  $\mathbf{C}$  is the whole of  $\mathbb{R}^3$ .

To find the null space of matrix  $\mathbf{C}$  we solve the equivalent system  $\mathbf{R}\mathbf{x} = \mathbf{0}$ :

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Expanding the bottom row we have  $w = 0$  and from the middle row we have  $z = 0$ . By expanding the top row we have

$$x + 3y = 0 \quad \text{gives} \quad x = -3y$$

Let  $y = s$  where  $s$  is any real number then  $x = -3y = -3s$ . Hence the general homogeneous solution is



$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -3s \\ s \\ 0 \\ 0 \end{pmatrix} = s \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Thus a basis for the null space of matrix  $\mathbf{C}$  is  $\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ . This gives  $\text{nullity}(\mathbf{C}) = 1$ .

6. (a) This is an undetermined system.

We first write out the augmented matrix and then evaluate the reduced row echelon form:

$$\begin{array}{cccc|c} x & y & z & w & \\ \hline 2 & 5 & 7 & 10 & 3 \\ 1 & 1 & 2 & 5 & 6 \end{array} \qquad \begin{array}{cccc|c} x & y & z & w & \\ \hline 1 & 0 & 1 & 5 & 9 \\ 0 & 1 & 1 & 0 & -3 \end{array}$$

The reduced row echelon form matrix is on the Right Hand Side. Expanding the bottom row of this Right Hand matrix we have

$$y + z = -3 \quad \text{which gives} \quad y = -3 - z$$

Expanding the top row yields

$$x + z + 5w = 9 \quad \text{which gives} \quad x = 9 - z - 5w$$

Let  $z = s$  and  $w = t$  then we have

$$y = -3 - z = -3 - s \quad \text{and} \quad x = 9 - z - 5w = 9 - s - 5t$$

The solution  $\mathbf{x}$  of the given non-homogeneous system is

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 9 - s - 5t \\ -3 - s \\ s \\ t \end{pmatrix} = s \underbrace{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}}_{=\mathbf{x}_H} + t \underbrace{\begin{pmatrix} -5 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{=\mathbf{x}_P} + \underbrace{\begin{pmatrix} 9 \\ -3 \\ 0 \\ 0 \end{pmatrix}}_{=\mathbf{x}_P}$$

Hence our solution is  $\mathbf{x} = \mathbf{x}_H + \mathbf{x}_P$  where  $\mathbf{x}_H$  is the homogeneous solution and  $\mathbf{x}_P$  the particular solution.

(b) The augmented matrix and its reduced row echelon form is given by

$$\begin{array}{ccc|c} x & y & z & \\ \hline 2 & -1 & -4 & 13 \\ 3 & 3 & -5 & 13 \\ 3 & -4 & 10 & -10 \end{array} \qquad \begin{array}{ccc|c} x & y & z & \\ \hline 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{array}$$

The matrix is of full rank. Reading off the rows of the Right Hand Matrix gives

$$x = 2, \quad y = -1 \quad \text{and} \quad z = -2$$

We have a unique solution and it is given by  $\mathbf{x} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$ .

7. We use the following Proposition in this question:

(3-26) (a)  $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A} \mid \mathbf{b}) = n$  then the linear system has a **unique** solution.

(b)  $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A} \mid \mathbf{b}) < n$  then an infinite number of solutions.

(c) If  $\text{rank}(\mathbf{A}) \neq \text{rank}(\mathbf{A} \mid \mathbf{b})$  then the linear system has **no** solution.

(a) Writing out the coefficient matrix,  $\mathbf{A}$ , and augmented matrix,  $(\mathbf{A} \mid \mathbf{b})$ , into reduced row echelon form we have

$$\mathbf{A} = \begin{pmatrix} 2 & 8 \\ 7 & 28 \end{pmatrix} \qquad \begin{pmatrix} 1 & 4 \\ 0 & 0 \end{pmatrix}$$

Putting the augmented matrix into reduced row echelon form we have

$$(\mathbf{A} \mid \mathbf{b}) = \left( \begin{array}{cc|c} 2 & 8 & 12 \\ 7 & 28 & 42 \end{array} \right) \qquad \left( \begin{array}{cc|c} 1 & 4 & 6 \\ 0 & 0 & 0 \end{array} \right)$$

What is the rank of  $\mathbf{A}$  and  $(\mathbf{A} \mid \mathbf{b})$  equal to?

In both cases we have  $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A} \mid \mathbf{b}) = 1$  because both matrices on the Right Hand Side have 1 non-zero row. Does the given system have any solutions?

Yes because we have 2 columns and  $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A} \mid \mathbf{b}) = 1 < 2$  therefore by

Proposition (3-26) part (b) we have an infinite number of solutions.

(b) Writing out the coefficient matrix,  $\mathbf{A}$ :

$$\mathbf{A} = \begin{pmatrix} 2 & -3 & -6 & 12 \\ 3 & -5 & -7 & 16 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & -9 & 12 \\ 0 & 1 & -4 & 4 \end{pmatrix}$$

Putting the augmented matrix into reduced row echelon form we have

$$(\mathbf{A} \mid \mathbf{b}) = \left( \begin{array}{cccc|c} 2 & -3 & -6 & 12 & 2 \\ 3 & -5 & -7 & 16 & 5 \end{array} \right) \qquad \left( \begin{array}{cccc|c} 1 & 0 & -9 & 12 & -5 \\ 0 & 1 & -4 & 4 & -4 \end{array} \right)$$

In both cases we have  $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A} \mid \mathbf{b}) = 2$  because both matrices on the Right Hand Side have 2 non-zero rows. Does the given system have any solutions?

Yes because we have 4 columns and  $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A} \mid \mathbf{b}) = 2 < 4$  therefore by

Proposition (3-26) part (b) we have an infinite number of solutions.

(c) Very similar to Example 39 part (b):

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rank}(\mathbf{A}) = 2$$

Putting the augmented matrix into reduced row echelon form we have

$$(\mathbf{A} \mid \mathbf{b}) = \left( \begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 9 & 4 \end{array} \right) \qquad \left( \begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{array} \right) \Rightarrow \text{rank}(\mathbf{A} \mid \mathbf{b}) = 3$$

We have  $\text{rank}(\mathbf{A}) = 2$  but  $\text{rank}(\mathbf{A} \mid \mathbf{b}) = 3$ . Since  $\text{rank}(\mathbf{A}) \neq \text{rank}(\mathbf{A} \mid \mathbf{b})$  so by Proposition (3-26) part (c) we conclude that the given system has **no** solution.

(d) Writing out the augmented matrix and placing into reduced row echelon form gives:

$$\left(\begin{array}{cccc|c} 2 & 5 & -3 & -7 & 0 \\ 1 & 1 & -4 & -8 & 9 \\ 3 & 4 & 0 & 1 & 6 \\ 5 & 21 & -1 & 3 & 2 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2.8376 \\ 0 & 1 & 0 & 0 & -1.5527 \\ 0 & 0 & 1 & 0 & -9.3248 \\ 0 & 0 & 0 & 1 & 3.6980 \end{array}\right)$$

This is the MATLAB output correct to 5sf.

What is the rank of  $\mathbf{A}$  and  $(\mathbf{A} \mid \mathbf{b})$  equal to?

In both cases we have  $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A} \mid \mathbf{b}) = 4$  because both sides of the vertical bar of the matrix on the Right Hand Side have no zero rows. Does the given system have any solutions?

Yes we have a **unique** solution because  $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A} \mid \mathbf{b}) = 4$  and the given system has 4 unknowns.

8. How do we check that a given vector is in the null space of a matrix?

Check that  $\mathbf{A}\mathbf{u} = \mathbf{0}$ .

(a) We have

$$\mathbf{A}\mathbf{u} = \begin{pmatrix} 1 & 1 & -1 \\ 2 & -1 & 0 \\ 5 & 2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{0}$$

Thus the given vector  $\mathbf{u}$  is in the null space of the matrix  $\mathbf{A}$ .

(b) Similarly we have

$$\mathbf{B}\mathbf{u} = \begin{pmatrix} 1 & 4 & -5 \\ -7 & 5 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$$

Hence the given vector  $\mathbf{u}$  is in the null space of the matrix  $\mathbf{B}$ .

(c) What do you notice about the multiplication  $\mathbf{C}\mathbf{u}$ ?

It is impossible because we have  $\mathbf{C}\mathbf{u} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$  where the number of columns of

matrix  $\mathbf{C}$  does **not** equal the number of rows of  $\mathbf{u}$ . Hence  $\mathbf{u}$  is **not** in the null space of matrix  $\mathbf{C}$ .

(d) We have

$$\mathbf{D}\mathbf{u} = \begin{pmatrix} 1 & -2 & 6 & 1 \\ 3 & -6 & 7 & 8 \\ 5 & 2 & 1 & 7 \\ 1 & 6 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ -4 \\ 7 \end{pmatrix} = \begin{pmatrix} -22 \\ * \\ * \\ * \end{pmatrix} \neq \mathbf{0}$$

The reason for placing a \* in the right hand vector is because we **cannot** have the zero vector since the first entry is  $-22$ . We do **not** need to evaluate the other entries.

The vector  $\mathbf{u}$  is **not** in the null space of matrix  $\mathbf{D}$ .

9. We need to prove that  $\mathbf{A}$  is invertible  $\Leftrightarrow \text{nullity}(\mathbf{A}) = 0$ .

*Proof.*

By applying Proposition (3-19) which says that:

$$\text{The matrix } \mathbf{A} \text{ is invertible} \Leftrightarrow \text{rank}(\mathbf{A}) = n$$

We have  $\text{rank}(\mathbf{A}) = n$ . By Theorem (3-22) which is

$$\text{nullity}(\mathbf{A}) + \text{rank}(\mathbf{A}) = n$$

Thus  $\text{rank}(\mathbf{A}) = n \Leftrightarrow \text{nullity}(\mathbf{A}) = 0$ . Hence we have our required result,  $\mathbf{A}$  is invertible  $\Leftrightarrow \text{nullity}(\mathbf{A}) = 0$ .

10. We need to prove that elementary row operations do **not** change the null space of a matrix.

*Proof.*

Let matrices  $\mathbf{A}$  and  $\mathbf{R}$  be row equivalent. By Proposition (1-11):

Proposition (1-11). If a linear system is described by the augmented matrix  $(\mathbf{A} \mid \mathbf{b})$  and it is row equivalent to  $(\mathbf{R} \mid \mathbf{b}')$  then both linear systems have the same solution set.

We have  $\mathbf{Ax} = \mathbf{0}$  and  $\mathbf{Rx} = \mathbf{0}$  have the same solution. Hence  $\mathbf{A}$  and  $\mathbf{R}$  have the same null space which means that elementary row operations do **not** change the null space.

11. We need to prove that every vector in the null space of matrix  $\mathbf{A}$  is orthogonal to every vector in the row space of matrix  $\mathbf{A}$  where  $\mathbf{A}$  is a  $m$  by  $n$  matrix.

*Proof.*

Remember the null space of matrix  $\mathbf{A}$  is the set of vectors  $\mathbf{x}$  which satisfy:

$$\mathbf{Ax} = \mathbf{0} \text{ where } \mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

Let  $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$  be a vector in the null space of matrix  $\mathbf{A}$ . What do we need to prove?

Required to prove that dot product of the row vectors  $\mathbf{r}$  of  $\mathbf{A}$  and  $\mathbf{x}$  is zero;  $\mathbf{r} \cdot \mathbf{x} = 0$ .

Carrying out the matrix multiplication  $\mathbf{Ax} = \mathbf{0}$  we have

$$\mathbf{Ax} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

For each row vector  $\mathbf{r}$  of matrix  $\mathbf{A}$  we have  $\mathbf{r} \cdot \mathbf{x} = 0$  therefore each row vector of matrix  $\mathbf{A}$  is orthogonal to the null space vector  $\mathbf{x}$ . This completes our proof.

12. We need to prove if  $\mathbf{A}$  is a matrix of size  $m$  by  $n$  then the null space of  $\mathbf{A}$  is a subspace of  $\mathbb{R}^n$ .

*Proof.* We know that null space is a member of  $\mathbb{R}^n$  because it is the set of vectors  $\mathbf{x}$  which satisfy  $\mathbf{Ax} = \mathbf{0}$  and vector  $\mathbf{x}$  has  $n$  entries.

How do we show that the null space is a subspace of  $\mathbb{R}^n$ ?

By using:

Proposition (3-5). A nonempty subset  $S$  is a subspace of a vector space  $V \Leftrightarrow \mathbf{u}$  and  $\mathbf{v}$  are vectors in  $S$  then any linear combination  $k\mathbf{u} + c\mathbf{v}$  is also in  $S$ .

Let vectors  $\mathbf{u}$  and  $\mathbf{v}$  be members of the null space of matrix  $\mathbf{A}$ . This means we have

$$\mathbf{A}\mathbf{u} = \mathbf{0} \text{ and } \mathbf{A}\mathbf{v} = \mathbf{0} \quad (\dagger)$$

Consider the linear combination  $k\mathbf{u} + c\mathbf{v}$ . What do we need to prove?

This vector  $k\mathbf{u} + c\mathbf{v}$  is also in the null space of matrix  $\mathbf{A}$ ; that is  $\mathbf{A}(k\mathbf{u} + c\mathbf{v}) = \mathbf{0}$ .

We have

$$\begin{aligned} \mathbf{A}(k\mathbf{u} + c\mathbf{v}) &= k(\mathbf{A}\mathbf{u}) + c(\mathbf{A}\mathbf{v}) \\ &= k(\mathbf{0}) + c(\mathbf{0}) \quad [\text{By } (\dagger)] \\ &= \mathbf{0} \end{aligned}$$

Hence by Proposition (3-5) we conclude that  $N$  is a subspace of  $\mathbb{R}^n$ .

13. We need to prove:

Proposition (3-24). The linear system  $\mathbf{Ax} = \mathbf{b}$  has a solution  $\Leftrightarrow \mathbf{b}$  is in the column space of matrix  $\mathbf{A}$ .

*Proof.*

Expanding out  $\mathbf{Ax} = \mathbf{b}$  where  $\mathbf{A}$  is a  $m$  by  $n$  matrix:

$$\begin{aligned} &\begin{matrix} \mathbf{c}_1 & \cdots & \mathbf{c}_n \end{matrix} \\ &\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \\ \Leftrightarrow &x_1 \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ \vdots \\ a_{m2} \end{pmatrix} + \cdots + x_n \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \quad \Leftrightarrow \quad x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n = \mathbf{b} \end{aligned}$$

Hence  $\mathbf{b}$  is in the column space of matrix  $\mathbf{A} \Leftrightarrow \mathbf{Ax} = \mathbf{b}$  has a solution.

14. We need to prove:

Proposition (3-26). Consider the linear system  $\mathbf{Ax} = \mathbf{b}$  where  $\mathbf{A}$  has  $n$  columns and  $\mathbf{b} \neq \mathbf{0}$ .

(a) If  $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A} \mid \mathbf{b}) = n$  then the linear system has a **unique** solution.

(b) If  $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A} \mid \mathbf{b}) < n$  then the linear system has an infinite number of solutions.

(c) If  $\text{rank}(\mathbf{A}) \neq \text{rank}(\mathbf{A} \mid \mathbf{b})$  then the linear system has **no** solution.

*Proof of (a).*

By question 8(b) of the last Exercise 3(e) which says:

$\mathbf{A}$  has rank  $n \Leftrightarrow$  the linear system  $\mathbf{Ax} = \mathbf{b}$  has a unique solution

We have if  $\text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A} \mid \mathbf{b}) = n$  then  $\mathbf{Ax} = \mathbf{b}$  has a unique solution. This is our required result.

*Proof of (b).*

Let  $r = \text{rank}(\mathbf{A})$  and we are given that  $r = \text{rank}(\mathbf{A}) = \text{rank}(\mathbf{A} \mid \mathbf{b}) < n$ .

We know by Theorem (3-22) that

$$\text{nullity}(\mathbf{A}) + r = n$$

From this we have  $\text{nullity}(\mathbf{A}) = n - r$  and since  $r < n$  therefore  $\text{nullity}(\mathbf{A}) > 0$ . Remember  $\text{nullity}(\mathbf{A}) = n - r$  is the dimension of the null space so let

$$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{n-r}\}$$

be a basis for the null space. Remember for any real scalars  $c_1, c_2, \dots, c_{n-r}$  the linear combination

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_{n-r} \mathbf{v}_{n-r} \quad (*)$$

is also in the null space of  $\mathbf{A}$ . Thus  $(*)$  is the solution of  $\mathbf{Ax} = \mathbf{b}$ , that is the homogeneous solution of  $\mathbf{Ax} = \mathbf{b}$ . Hence we have an infinite number of solutions because the result is true for an infinite number of scalars.

*Proof of (c).*

Already proven in Proposition (3-25) of the main text.